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LETTER TO THE EDITOR

Hierarchy of effective sizes of DLA clusters

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Abstract. We discuss the possibility of having an infinite hierarchy of length scales characterising the moments of the Laplacian field outside DLA clusters, *in the entire space.* This hierarchy constitutes a multifractality for DLA which is different from the usual one associated with the near field of these clusters. The radius of gyration and the hydrodynamic radius are both members of this new family of length scales. We present numerical simulations on the usual DLA model, and on the fracture growth model (i.e. 'elastic' DLA), testing this hypothesis.

Most of the earlier investigations of stochastic growth models, such as the DLA model [1], concentrated on purely *geometrical* properties of the growing clusters (see especially the first reference of [1]). However, during the last two years, several studies of some dynamical aspects have appeared [2]. In particular, the growth probability on the surface of these clusters has shown a very rich structure known as multifractality [3]. This means that the moments of the probability distribution scale with an infinite set of independent exponents. This phenomenon has several practical consequences, such as the fact that the drag force distribution on the surface of the clusters in a flow is anomalously broad [4]. The growth models are all based on the interaction between the growing surface and a surrounding Laplacian potential ϕ. The multifractality of the growth probability distribution is a consequence of the behaviour of the gradient of the field, $\nabla\phi$, at the surface of the clusters. However, one may ask whether there is some non-trivial behaviour of this gradient also in the *bulk* surrounding the growing clusters. Cates and Witten suggest that no such bulk multifractality exists [5]. However, this is certainly an important question, as the interactions of the cluster as a whole with its surroundings are determined by the bulk behaviour of ϕ. Testing this is the aim of this letter. It is easy to show that the zeroth and infinite moments are different. A similar argument suggests that there is a discontinuity at the zeroth moment, and all the other scaling exponents are equal. Whether this is so, or whether there is a 'genuine' multifractality with an infinite set of different exponents, is not possible to determine from the computer simulations. The multifractality of the bulk distribution in connection with growth models may be expressed as an infinite hierarchy of length scales describing the effect of the clusters on the surrounding Laplacian field. The radius of gyration of the clusters is associated with the scaling of the infinite moment,

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and the hydrodynamic radius with the second moment. Our numerical results are based on numerical studies of the dual of the usual diffusion-limited aggregation (DLA) model, and the fracture growth model [6, 7] which is a central-force elastic version of the dual of the DLA model.

We briefly recall the definition of DLA, and its dual, the DDLA model. Although the two problems are equivalent in two dimensions, this is not the case in higher dimensions of space. The concept of duality in connection with the DLA model makes clear the connection with fracture, or more precisely, the fracture growth model (FGM).

We first consider the case of DLA. For simplicity, we will only consider here the case of a cluster grown on a lattice. Let us consider a single bond that will constitute the initial seed of the cluster. All bonds belonging to the growing cluster are given a uniform zero potential. At infinity the potential is assumed to be unity. In the space which is not occupied by the DLA cluster, we compute the harmonic potential ϕ which satisfies the Laplacian equation $\nabla^2\phi = 0$, and the previous boundary conditions. Next, we grow the cluster by adding to it a bond that was lying on its perimeter. This bond is selected at random, with a probability proportional to the local potential difference across the bond. Then the Laplace equation is solved again with the new boundary conditions arising from the modification of the geometry of the cluster. A new perimeter bond is then chosen according to the previously described algorithm, and subsequently added to the cluster. This is the most classical geometry used to study DLA. We can, however, consider other types of boundary conditions that will not change the observable scaling properties when the clusters do not come too close to the boundaries of the lattice. For instance figure 1(*a*) shows the implementation of this procedure in a finite square box, where a whole side of the square was chosen to be the seed of the cluster. These boundary conditions simplify the connection with the DDLA model, to which we now turn.

We define the dual DLA (DDLA) model by replacing potential gradients by currents and insulating borders by equipotential ones and vice versa, following the method of, e.g., Straley [8]. After this substitution, the growing cluster consists of insulating material. One way of visualising this duality transformation is to think of a cluster of broken bonds in a resistor network. The Laplace equation is unchanged in the rest of the medium; see figure 1(*b*). The growth probability on the surface of the growing cluster is now proportional to the current flowing along it, instead of the potential gradient orthogonal to the surface. In two dimensions the original DLA model and the dual one are equivalent. This can be shown in the continuum limit by considering the holomorphic complex function whose real part is the potential ϕ and whose imaginary part is the potential distribution of the corresponding DDLA problem. This correspondence also works directly for the lattice problem, as was shown by Straley [8].

The dual DLA problem is already close to a fracture problem. If the cluster is grown on a lattice, the bonds that become part of this growing cluster turn from being resistors into insulators. This growth can thus be seen as a process where fuses on the surface of a cluster of already burnt-out fuses break with a probability proportional to the current flowing through them. Now, if we turn this problem into an elastic one by replacing the electrical fuses by elastic elements that break if the stress they carry exceeds some critical limit, we have a model [6, 7] for important rupture processes such as stress-induced corrosion. One may state this conversion from an electrical problem into an elastic one in a more general way, however: the only change that needs to be done in order to turn the DDLA model into an elastic growth model is to incorporate the operator of linear elasticity that corresponds to the Laplacian [9]. One

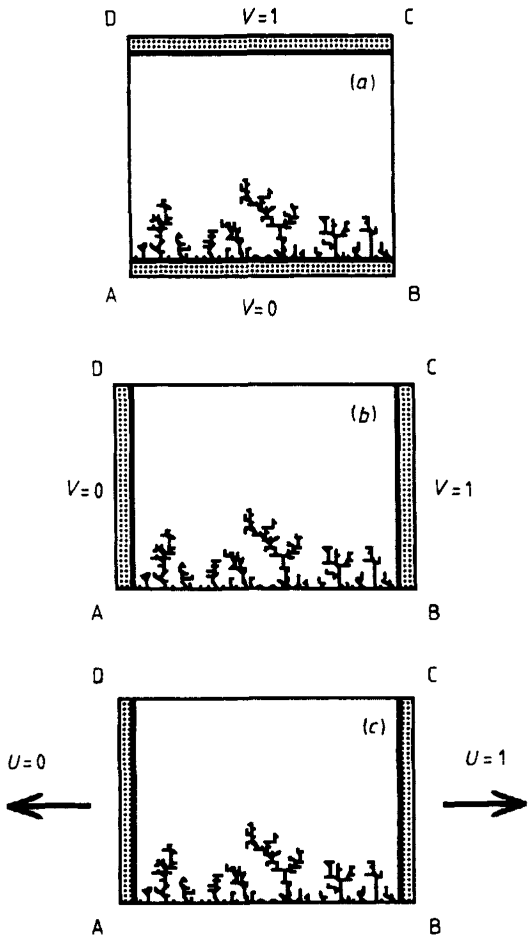

Figure 1. (*a*) A particular set of boundary conditions which allows us to grow DLA clusters from the basis line AB where all sites, including those of the DLA structures, are set to a zero potential. The upper line CD is at a constant potential of 1. The two border sides, AD and BC, are insulating. The DLA is grown by choosing at random a site on its boundary, with a probability proportional to the local potential.

(*b*) The dual DLA model is constructed from the previous case by exchanging the role of currents with potential gradients, and equipotential parts with insulators. Now the DLA cluster is insulating, and the growth probability is proportional to the current passing through a neighbouring bond to the cluster on the dual lattice. Although the DDLA model is strictly equivalent to DLA in two dimensions this is not the case in three (and higher) dimensions.

(*c*) The fracture growth model is obtained from the previous DDLA by changing the Laplace operator into the Lamé one (see text for definition). The potential is turned into the displacement field, and the growth probability is proportional to the absolute value of the force. The displacement is imposed on the two lateral edges AD and BC. The fracture cluster is infinitely soft.

way of implementing this on a lattice is to work with a central-force lattice. In this model, known as the fracture growth model (FGM), the bonds consist of linear Hookean springs free to rotate about their endpoints. A given displacement is imposed on the boundary as shown in figure 1(*c*). Then a spring is broken on the border of the existing cluster—or crack—with a probability proportional to the force carried by the bond. We now investigate the scaling of the field outside the clusters.

The solution of Laplace's and the elastic equations for the case of a circular hole has the form

$$\phi(r) = \phi_0(r) + Ar^{-(d-1)} \tag{1}$$

where $\phi_0(r)$ is the potential (or displacement) for the medium without the hole, for boundary conditions expressed in terms of the field, $\nabla\phi$. The nth moment of the field gradient relative to that of the background field ϕ_0 is, for n positive,

$$\langle[\nabla\phi(\rho)]^n\rangle_l - \langle[\nabla\phi_0(\rho)]^n\rangle_l = \int_R \mathrm{d}\rho\, \rho^{d-1}\{[\nabla\phi(\rho)]^n - [\nabla\phi_0(\rho)]^n\} \propto R^{d(1-n)} \tag{2}$$

where R is the radius of the hole. If now, instead of keeping the field constant at infinity, one chooses to keep the potential fixed, the scaling becomes

$$\langle[\nabla\phi(\rho)]^n\rangle_V-\langle[\nabla\phi_0(\rho)]^n\rangle_V=\int_R \mathrm{d}\rho\,\rho^{d-1}\{[\nabla\phi(\rho)]^n-[\nabla\phi_0(\rho)]^n\}\propto R^d. \qquad (2')$$

These properties can be used to define effectives sizes of DLA clusters, estimated on various measures (moments) of the external field.

Let us compute the nth moment, m_n, of the potential gradient *outside the DLA cluster* as defined by the LHS of (2′), in the constant potential ensemble. This moment scales with the mass, M, of the cluster according to the power law

$$m_n \propto M^{y(n)}. \qquad (3)$$

Comparing the scaling with the one obtained for a circular hole of radius R, one can define a hierarchy of effective radii R_n, by equating the two expressions:

$$R_n \propto M^{y(n)/d}. \qquad (4)$$

We thus find a relation between the radius R_n and the mass, which obviously depends on the measure used, or equivalently on n. When $n=0$, then $y(0)=1$, and R_0 is the radius of a circular hole whose area is equal to that of the DLA cluster. Here, the moments may behave differently in the limit $n\to 0^+$, as for $n=0$. This situation may occur if the screening of the field ϕ within the fjords of the DLA cluster is strong enough (i.e. exponentially damped) so that the cluster looks compact from the outside. If this is so, then $y(n\to 0^+)=d/d_f$, and there is a discontinuity of the exponents at the point $n=0$. Using the geometrical radius of gyration R_g of the DLA, which scales as $M\propto R_g^{d_f}$, we obtain

$$R_0 \propto R_g^{d_f/d}. \qquad (5)$$

When $n=2$, R_2 is the radius of a circular hole which reduces the conductivity of a plane as much as the cluster. Due to screening, one expects the scaling exponent of R_2 to be larger than that of R_0. Indeed, figures 2(a) and 2(b) show the log-log plot of three moments m_1, m_2 and m_3 against the mass of the cluster, from which we can extract the three exponents $y(1)$, $y(2)$ and $y(3)$ reported in table 1. Here, for DDLA, $y(2)=1.2\pm 0.1$. From this it follows that

$$R_2 \propto R_g^{y(2)d_f/d} \propto R_g^{z(2)} \qquad (6)$$

with $z(2)=1.0\pm 0.1$. This exponent is close to 1, indicating that the conductivity radius is very close to the radius of gyration. A similar radius would tell us the change in the viscosity of the flow outside the cluster due to its presence if this is interpreted as a hydrodynamic problem. The $n=4$ moment may be related to the second moment of the fluctuations of the conductivity of the medium surrounding the hole.

For large-order moments, the 'hottest' part of the cluster neighbourhood will be weighted dominantly. Therefore, the sites that will contribute most to these large-order moments will be located near the tips of the DLA cluster, and located far from its centre. However, since the extreme extent of the cluster scales the same way as the radius of gyration, as shown by recent numerical simulations [10], we expect $\{y(n)d_f/d\}$ to approach 1 when n goes to infinity, or $y(n)$ to tend to d/d_f (=1.17 in 2D for DLA).

We studied numerically two problems: the DDLA and the FGM. For both cases, we generated square-shaped 64×64 triangular lattices. For DDLA, the top and bottom rows were bus bars set to a fixed potential of 0 and 1 respectively. We implemented

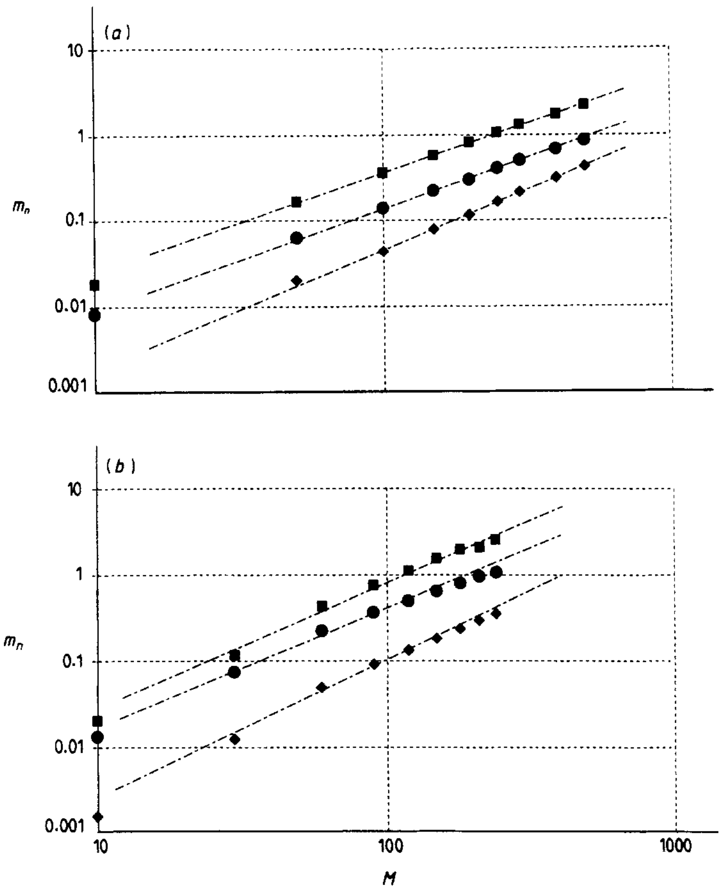

Figure 2. (*a*) Log-log plot of the first three moments of the current distribution outside a DDLA cluster against its mass compared with the intact lattice. (*b*) Log-log plot of the first three moments of the force distribution outside a crack (FGM model) against its mass compared with the intact lattice. In both parts, plots are shown for $n=1$ (◆), $n=2$ (●) and $n=3$ (■).

Table 1. Values of the different exponents $y(n)$ and $z(n)$ for $n=0$-3, for both models studied in this letter. The data are extracted from the plot of the evolution of the moments against the mass M, for large M, as shown in figure 2(*a*) for the DDLA model and in figure 2(*b*) for the FGM model.

Moment order	DDLA		FGM
n	$y(n)$	$z(n)$	$y(n)$
0	1	0.85	1
1	1.3	1.1	1.6
2	1.2	1.0	1.5
3	1.2	1.0	1.5

periodic boundary conditions on the other edges. The cluster was initiated by the removal of the central bond of the lattice. The FGM was treated in a similar manner. In this problem, each bond is a linear spring which can rotate at its endpoints. We imposed a displacement on the two bus bars (now infinitely rigid): one was fixed and the other was sheared (displacement along its axis), or compressed (displacement perpendicular to it). A first bond broken in the centre of the plane constitutes the starting stage of the cluster (i.e. the crack).

For both problems, we computed the potential or the displacement outside the cluster by a conjugate gradient technique [11], with using the norm of the residual vector as stopping criterion, and chosen to be 10^{-10}. A more detailed presentation of the computation has been presented in an earlier study [7]. We invested about 25 CPU hours on a Cray X-MP on these calculations.

The data extracted from our numerical simulation show a surprising fact: the exponents $y(n)$ obtained for both the vector and the scalar case start for $n=0$ by $y(0)=1$, as per definition. Then the value of $y(n)$ increases sharply for $n=1$, and then decreases for subsequent values of n toward their asymptotic value d/d_f when the exponents are extracted for small values of M. This non-monotonicity of the evolution of $y(n)$ is counter-intuitive if we compare it with the usual behaviour of the infinite hierarchy of exponents appearing in multifractal phenomena. Two competing factors are responsible for this behaviour. (i) The higher the order of the moment, n, the less weight is given to the inner 'fjords' screened by the arms of the DLA. This causes an increase of $y(n)$ with n. (ii) For low-order moments, the outside of the cluster also has a large weight, and therefore even regions located far from the cluster contribute significantly to the moments. If we consider only this trend, then the exponent $y(n)$ should decrease with n. This latter argument may appear also as an enhancement factor of the spurious edge effects necessarily present at a finite distance in our approach.

Indeed, one can show easily that the second exponent $y(2)$ should be less than or equal to the limit $y(\infty)=d/d_f$. If one cuts out of the plane a circular hole that contains the cluster, then the energy (second moment) will decrease, and since the radius of the hole is proportional to the radius of gyration, we derive the bound $y(2)\leq d/d_f$.

A recent numerical study of the hydrodynamic radius of DLA clusters in three dimensions, by Chen *et al* [12], has suggested that the latter radius increases faster than the gyration radius in contrast to bond-percolation clusters, where both radii seem proportional. For another kind of fractal aggregate studied experimentally by Wiltzius, i.e. colloidal silica aggregates [13], both radii seem proportional.

We are thus tempted to conclude that the trend observed in our numerical simulations, i.e. the fact that $y(1)$ is larger than d/d_f, is only a finite-size effect. However, the fact that $y(2)$ and $y(3)$ are already very close to d/d_f is intriguing. If $y(2)$ is equal to d/d_f, we expect naturally that for all n larger than 2, $y(n)$ is also equal to d/d_f. However, it is clear that $y(0)=1$. As suggested above, there might be a discontinuity in the $y(n)$ spectrum between the value of $y(0)$ and $y(n\to 0^+)$ in the case of exponentially strong screening. Therefore, between $n=0$ and 2, there is either a continuous increase of $y(n)$ or an abrupt jump. The first case would thus restore the infinite hierarchy of effective sizes of the clusters, whereas the second would imply that the hierarchy of exponents contains two discrete singularities that can be isolated by a multifractal analysis. We note that the situation for $n<2$ is special already for the simple two-dimensional case of a circular hole in a restricted geometry such as the one studied here. By solving the Laplace equations analytically for this case, one finds that the first moment scales as R^2, where R is the radius of the hole. This result is counter-intuitive, since simple power counting in (3) indicates that this moment should be governed by the *outer* boundary, and not by R. Clearly, some delicate balancing of effects are in play for this moment.

In any case, careful numerical studies of this question would prove useful in describing these various effective sizes of DLA clusters and the eventual existence of an infinite hierarchy, or of a discrete spectrum of singularities. We regard our numerical result as only suggestive, and urge further such studies of this interesting question, but warn that they require huge amounts of computation time.

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References

[1] Meakin P 1988 *Phase Transitions and Critical Phenomena* vol 12 ed C Domb and J L Leibowitz (New York: Academic) p 335
Feder J 1988 *Fractals* (New York: Plenum)
[2] Måløy K J, Boger F, Feder J, Jøssang T and Meakin P 1987 *Phys. Rev.* A **36** 318
Pietronero L, Erzan A and Evertsz L 1988 *Phys. Rev. Lett.* **61** 861
Alstrom P, Trunfio P and Stanley H E 1989 *Fluctuations and Pattern Formation* ed H E Stanley and N Ostrowsky to appear
[3] Amitrano C, Coniglio A and di Liberto F 1986 *Phys. Rev. Lett.* **57** 1016
Pietronero L and Siebesma P 1986 *Phys. Rev. Lett.* **57** 1098
Meakin P 1985 *J. Phys. A: Math. Gen.* **18** L661
[4] Witten T A 1987 *Phys. Rev. Lett.* **59** 901
[5] Cates M E and Witten T A 1987 *Phys. Rev.* A **35** 1809
[6] Louis E and Guinea F 1987 *Europhys. Lett.* 3 871
[7] Hinrichsen E L, Hansen A and Roux S 1989 *Europhys. Lett.* **8** 1
[8] Straley J P 1977 *Phys. Rev.* B **15** 5733
[9] Landau L D and Lifshitz E M 1967 *Théorie de l'Elasticité* (Moscow: MIR)
[10] Hinrichsen E L, Mäløy K J, Feder J and Jøssang T 1989 *J. Phys. A: Math. Gen.* **22** L271
Argoul F, Arneodo A, Grasseau G and Swinney H L 1988 *Phys. Rev. Lett.* **61** 2558
[11] Batrouni G G and Hansen A 1988 *J. Stat. Phys.* **52** 747
[12] Chen Z Y, Deutch J M and Meakin P 1984 *J. Chem. Phys.* **80** 2982
Chen Z Y, Weakliem P C and Meakin P 1989 *J. Chem. Phys.* to appear
[13] Wiltzius P 1987 *Phys. Rev. Lett.* **58** 710
Wiltzius P and van Saarloos W 1987 *Phys. Rev. Lett.* **59** 2123